Hilbert's Sixth Problem and Topos Theory

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> `\There are more things in heaven and earth, Horatio, Than are dreamt of in your philosophy."`

This talk will cover one particular aspect of the yet unsolved Hilbert's Sixth Problem (HSP) from a category theoretic point of view[1]. This is one of the 23 problems Hilbert proposed to the then community of mathematicians in Paris at the turn of the 20th century and are largely responsible for shaping mathematics of the last century. HSP called out for an axiomatization of physics and was, therefore, a programmatic call. Given that HSP has had well over a century to develop, and that the question was a rather broad one, it has initiated a long conversation between mathematics, physics and philosophy. It is, therefore, impossible to talk about all the major developments of HSP in 50 minutes but I will try to convey the flavour of why this problem is important for the development of both math and physics together. For special issues dedicated to the evolution and current status of HSP, see [3][1]. For a detailed introduction to the topic of this talk, including possible future directions of research see [4]. For a briefer introduction, see [6], which is a chapter in the fancy sounding [5].

There is a long list of unsolved problems in physics. A brief visit even to its Wikipedia page should convince you. Some of these talk about violations of parity, some talk about a 55 orders magnitude difference in theory and experimental observation, others talk about no violations of time-symmetry, which is required in order to explain the matter-antimatter discrepancy, while some puzzle over why the arrow of time is the way it is. Conversely, the reification of theoretical observations is resting on empirical confirmations. According to Lee Smolin, the situation is embarrassing[10].

All such details are based on two frameworks of physics namely a theory of the very big and a theory of the very small – and there aren't just two! For example, David Bohm came up with his own version of the theory of the atomic world, called Pilot Wave Theory in agreement with Quantum Mechanics and Bell's Theorem[2]. Other theoretical variations of a theory of gravity, of which there is no experimental confirmation, yet, are Double Special Relativity, de Sitter Relativity and Loop Quantum Gravity.

[1] An alternate appraoch, also using category theory, involves symmetric monoidal dagger-categories, as a generalization of Hilbert Spaces. For a toy model, see [11]. This is not what we will be talking about.

[2] No non-local, hidden variable theory can agree with the predictions of quantum mechanics

Which theory should trump? Of course the one nature thinks is right. Given all current tests performed so far, both the relativity of Einstein (herein after called GR) and Quantum Mechanics (QM) have stood up to scrutiny. For example, the (weak) equivalence of gravitational and inertial mass has been tested up to 15 decimal places by MICROSCOPE but no experiment has been performed using antimatter. A voilation, if it should exist, is expected in between 10^{-13} and 10^{-18} range. Does light travel at the same speed in a vacuum, regardless of wavelength? What about ultra-high energy gamma waves, which have a wavelength of 1.24×10^{-20} m? At such scales, the effects of quantum fluctuations might come into play.

Since the most successfully tested theories, to date, are GR and QM, we shall focus on these frameworks and talk about the mathematical incompatibility of the two and possible common language for the two. For empirical details and justification on this narrowed focus, see the comprehensive [9].

1 Differences in Mathematical Framework of GR and QM

A physical theory requires a space of states with equations that govern how the states evolve. There are physical quantities assigned to states and their properties are may be described by their values. For example, the governing equation of a pendulum convey information (state) of velocity (physical quantity) of the pendulum with particular magnitude (property). These properties can be described in terms of yes-no questions. For example, "is the velocity $4ms^{-1}$", or "is the mometum between $-5kgms^{-1}$ and $5kgms^{-1}$"?

For GR, a system S is modelled by a 4-dimensional smooth[3] manifold M for a local coordinate system, hence the term "relativity". M, required to be the set of states, is equipped with an everywhere nondegenerate[4], bilinear from (a metric tensor) $\varphi : T_p(M) \times T_p(M) \longrightarrow \mathbb{R}$ where $T_p(M)$ is the tangent space of the manifold for each state $p \in M$ with signature $(3, 1)$. That is, $\varphi(\mathbf{x}, \mathbf{y}) = x_1 y_1 + x_2 y_2 + x_3 y_3 - x_4 y_4$. This bilinear form is not required to be postive definite[5] and the absence of positive definiteness allows us to have a "metric" with negative values. Hence what we have is not a Riemannian manifold but a pseudo-Riemannian manifold. Since what we have is a differentiable manifold, we are allowed to consider continous functions $\chi : M \longrightarrow \mathbb{R}$. Had M been a (pseudo-)Hermitian Manifold, we could have considered functions $\chi : M \longrightarrow \mathbb{C}$ but at any rate, such functions are considered to represent a physical quantity x. Properties of x are restricted to Borel subsets E of $\mathbb{R}$ (or $\mathbb{C}$). Thus, the

[3] The requirement for smoothness is a mathematical necessity. For example, in defining a metric, one needs smooth functions γ for the integral in $d(x,y) = \inf\left\{\int_0^1 \|\gamma'(t)\|\, dt : \gamma : [a,b] \longrightarrow M, \gamma(a) = x, \gamma(b) = y\right\}$ to make sense.

[4] $y \widetilde{\longmapsto} (x \longmapsto \varphi(x,y))$

[5] $\varphi(\mathbf{x}, \mathbf{x}) > 0$ for $\mathbf{x} \neq 0$. This is opposite to the idea of isotropy: vectors $\mathbf{x}$ with $\varphi(\mathbf{x}, \mathbf{x}) = 0$ are said to be isotropic.

physical quantity x is said to have property E, which we shall write as $x \vdash E$, if for a state $p \in \chi^{-1}(E)$, $\chi(p) \in E$.

On the other hand, for QM, one needs a Hilbert Space H, very special manifold, to model a physical system. A closed subspace of H corresponds to a quantum state of a given system. Physical quantities, called observables, are modelled by a special class of operators on H. If χ is the observable corresponding to physical quantity x, $x \vdash E \iff E \subset Sp(\chi)$.

Obviously, the function $\varphi : H \times H \longrightarrow \mathbb{R}$ for a Hilbert Space is not bilinear (it's sesquilinear) but is definitely positive definite. It turns out that this difference is the biggest and the beauty of the axiomatic method tells us that we need to look no further to find common grounds. We outline why.

Definition 1 *Let X be a vector space over $\mathbb{K}$. A **f-sesquilinear 2-form** is a function $\varphi : X \times X \longrightarrow \mathbb{K}$ such that $\forall \alpha \in \mathbb{K}$ and $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in X$,*

S1 $\varphi(\mathbf{x}+\mathbf{y}, \mathbf{z}) = \varphi(\mathbf{x}, \mathbf{z}) + \varphi(\mathbf{y}, \mathbf{z})$

S2 $\varphi(\mathbf{x}, \mathbf{y}+\mathbf{z}) = \varphi(\mathbf{x}, \mathbf{y}) + \varphi(\mathbf{x}, \mathbf{z})$

S3 $\varphi(\alpha\mathbf{x}, \mathbf{y}) = f(\alpha)\varphi(\mathbf{x}, \mathbf{y})$

S4 $\varphi(\mathbf{x}, \alpha\mathbf{y}) = \varphi(\mathbf{x}, \mathbf{y})\alpha$

where $f : \mathbb{K} \longrightarrow \mathbb{K}$ is an involutive anti-automorphism. We shall call the tuple $(X, \mathbb{K}, \varphi, f)$ a sesquilinear space.

It is easy to see that $\varphi(\mathbf{0}, \mathbf{y}) = \varphi(\mathbf{x}, \mathbf{0}) = 0$ because $\varphi(\mathbf{0}, \mathbf{y}) = \varphi(\mathbf{x}-\mathbf{x}, \mathbf{y}) = \varphi(\mathbf{x}, \mathbf{y}) - \varphi(\mathbf{x}, \mathbf{y}) = 0$ and so, we can define the orthogonal complement $A^{\perp}$ of a set of elements A. This allows us to define a closure operator $A \longmapsto A^{\perp\perp}$ in the sense of [7] which converts an ordinary set into a subspace by respecting containment of the set, monotonicity and idempotency. Let $(X, \mathbb{K}, \varphi, f)$ be a sesquilinear space with $f(\mathbf{x}) = -\mathbf{x}$ (that is, φ is antisymmetric). Using a particular case of this formalism, it has been shown in [8] that if φ admits nonzero isotropic vectors, then there are closed subspaces of X that are not splitting. Equivalently, if X is orthomodular[6], then X has no nonzero isotropic vectors. Hilbert Spaces have this property whereas the tangent space of a pseudo-Riemannian manifold does not.

From these geometric arguments, we move to algebraic arguments.

Definition 2 *Let $\mathcal{L}$ be a non-empty set. Then, a **lattice** is an algebraic structure $(\mathcal{L}, \wedge, \vee)$ with two binary operations "$\vee$" and "$\wedge$", called join and meet, respectively, such that for all $x, y, z \in \mathcal{L}$*

1. $x \vee x = x$ *(idempotent w.r.t $\vee$)*
2. $x \wedge x = x$ *(idempotent w.r.t $\wedge$)*

[6] X is orthomodular if for all closed $F \subseteq X$, $X = F \oplus F^{\perp}$

3. $x \vee y = y \vee x$ *(commutative w.r.t* $\wedge$*)*

4. $y \wedge x = x \wedge y$ *(commutative w.r.t* $\vee$*)*

5. $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ *(associative w.r.t* $\wedge$*)*

6. $(x \vee y) \vee z = x \vee (y \vee z)$ *(associative w.r.t* $\vee$*)*

7. $x \wedge (x \vee y) = x$ *(meet-absorption law)*

8. $x \vee (x \wedge y) = x$ *(join-absorption law)*

With a lattice, we can easily move back and forth to a partial order using $x \leq y \iff x \vee y = x$, or, equivalently, via $x \leq y \iff x \wedge y = y$. A lattice $\mathcal{L}, \wedge, \vee)$ is said to be **bounded** if there exists elements $0, 1 \in \mathcal{L}$ such that $0 \leq x \leq 1$ for all x. For a bounded lattice and $x \in \mathcal{L}$, the element $x' \in \mathcal{L}$ such that $x \wedge x' = 0$ and $x \vee x' = 1$ is said to be the complement of x. $\mathcal{L}$ is said to be **complemented** if every element has a complement. A lattice is called **distributive** if either $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ or $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \mathcal{L}$ is said to be **orthocomplemented** if there is a function $f : \mathcal{L} \longrightarrow C$, called an **orthocomplementation**, such that

1. $f(x) \in C_x$
2. $f(f(x)) = x$
3. $x \leq y \implies f(y) \leq f(x)$ for all $x, y \in \mathcal{L}$

where C is the set of complements of $\mathcal{L}$ and C_x is the set of complements of x. A Boolean Lattice is an orthocomplemented, distributive lattice. Stone's Representation Theorem tells us that for every Boolean algebra (lattice) is isomorphic to $(\mathcal{P}(X), \cup \cap, ^c)$. For reasons that evade me[7], a collection of objects induced by a pseudo-Riemannian manifold form a Boolean Algebra.

On the other hand, a lattice $\mathcal{L}$ is **orthomodular** if $x \leq z$ implies $x \vee (x' \wedge z) = z$ for all $x, z \in \mathcal{L}$. It is an easy exercise to show that $(X, \mathbb{K}, \varphi, f)$ is orthomodular if and only if the lattice formed by a collection of closed subspaces of X is orthomodular, hence the name. A Boolean Lattice models classical logic whereas orthomodular lattice models Quantum Logic, with implication $x \Longrightarrow y$ in both given by $x \leq y$.

A distributive lattice is orthomodular but the converse does not hold[8]. The collection of closed subsets of a Hilbert space H form an orthomodular lattice, with complement given by the perp operator (collection of elements orthogonal to vectors in a given set), meet and join defined by intersection and direct

[7]This has something to do with what's called *measurable locale*. These require that the manifold be smooth and force us to consider only Borel measurable sets instead of the richer Lebesgue measurable sets

[8]In fact, a formulation of the Kochen-Specker tells us that any orthomodular lattice will admit a nontrivial morphism to $\{0, 1\}$ only if it is distributive.

sum, respectively. This lattice is not distributive: for $H = \mathbb{R}^2$, consider subspaces $X = (x, 0)$, $Y = (0, y)$ and $Z = (x, mx)$. Then, $(X \oplus Y) \cap Z = Z$ but $(X \cap Z) \oplus (Y \cap Z) = \{0\}$. Physically, there is a nice explanation of why distributive law fails, based on the Hiesenberg Uncertainty Principle (and this follows simply from the noncommutativity of two operators) on the Wikipedia Page for Quantum Logic.

The differences between the two major frameworks of physics are, therefore, inherent in the way they have been set up and a unification for them requires a departure from either tool.

2 Enter Category Theory

In general, a category inherently lacks precise statements about its objects and is more focused on the relationships between the objects. This shortcoming is a little refined in a topos, which is simply a category $\mathcal{C}$ with finite limits and colimits, exponentials and a subobject classifier. The category `Set` is an example, as is the category of presheaves of set on $\mathcal{D}$ $\mathtt{Set}^{\mathcal{D}^{op}}$.

Recall that a **limit** L in $\mathcal{C}$ is the datum given by an indexing category $\mathbf{I}$, a covariant functor $\mathcal{F} : \mathbf{I} \longrightarrow \mathcal{C}$ such that for each $\alpha \in Hom_{\mathbf{I}}(I, J)$, we have morphism $\lambda_I \in Hom_{\mathcal{C}}(L, I)$, $\lambda_J \in Hom_{\mathcal{C}}(L, J)$ with $\lambda_J = \mathcal{F}(\alpha) \circ \lambda_I$

$$\begin{tikzcd} & L \arrow[dl, "\lambda_I"'] \arrow[dr, "\lambda_J"] & \\ \mathcal{F}(I) \arrow[rr, "\mathcal{F}(\alpha)"] & & \mathcal{F}(J) \end{tikzcd}$$

and that L is final with this property. With finite limits and colimits, we can talk about products and coproducts, terminal objects, equalizers and coequalizers (e.g. kernels and cokernels in `Ab`). With the terminal object T, we can define an "element" (say x) of an object X (no matter what X is) by the morphism $\widehat{x} : T \longrightarrow X$. This is technically called a **global element** of an object X. Thus, the collection of global elements of X is $Hom_{\mathcal{C}}(T, X)$.

Exponentiation for objects Y, Z of $\mathcal{C}$ is defined as the object Z^Y together with a morphism $f : Z^Y \times Y \longrightarrow Z$, if for any object X and morphism $g : X \times Y \longrightarrow Z$, there is a unique morphism $\vartheta : X \longrightarrow Z^Y$ such that

$$\begin{tikzcd} X \times Y \arrow[d, "\vartheta \times \mathrm{id}_Y"'] \arrow[dr, "g"] & \\ Z^Y \times Y \arrow[r, "f"] & Z \end{tikzcd}$$

In this case, ϑ represents curried g. Think of a homotopy $g : [0, 1] \times Y \longrightarrow Z$ given by $g(t, .) = g_t(.) = \vartheta(t)$. This example, hopefully, makes it clear that g is called an evaluation map (ϑ is called **transpose** of g). Exponentiation allows us to talk about morphisms but in a more general setting since $Hom_{\mathcal{C}}(X \times Y, Z) \simeq Hom_{\mathcal{C}}(X, Z^Y)$.

A **subobject classifier** is a special object Ω such that for any set A, an object $f \in Hom_{\mathcal{C}}(A, \Omega)$ corresponds bijectively to a subobject of A. Thus, it allows us to talk about subojects of objects which are not sets. For sets, this is $\Omega = \{0, 1\}$ and $f = \chi_S$ for some $S \subset A$. Thus, the collection of subobjects of an object X is none other than $Hom_{\mathcal{C}}(X, \Omega)$. In fact, with exponentiation, we are allowed to talk about Ω^X, called the **power object** of X. The collection of subjects of Ω^X is, by definition, $Hom_{\mathcal{C}}\left(T, \Omega^X\right) \simeq Hom_{\mathcal{C}}(T \times X, \Omega) \simeq Hom_{\mathcal{C}}(X, \Omega)$, which is nothing but the collection of subobjects of X.

Since we have $Hom_{\mathcal{C}}(T, X)$ and $Hom_{\mathcal{C}}(X, \Omega)$ and we are allowed to compose morphisms in any category, we can talk about a global element x being in a subobject K of an object X since we can consider the morphism $\chi_K \circ \widehat{x} : \{*\} \longrightarrow X \longrightarrow \Omega$. This allows us to talk about valuations.

A Heyting Algebra $\mathcal{H}$ is a bounded, distributive lattice with a binary operation $\underline{\rightarrow}: \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{H}$ such that $c \wedge a \leq b$ if and only if $c \leq a \underline{\rightarrow} b$. A Boolean lattice is an example if we define complement a' of an element a by $a \underline{\rightarrow} 0$. Heyting Algebras model intuitionistic logic. Thus, the law of excluded middle does not hold in a Heyting Algebra. The collection Ω^X of subobjects of X form a Heyting Algebra.

3 Motivation the Physics for Topos

The opening of last section talked about the requirements for a physical theory. These distill down to signalling a special object for the state x, a morphism for the physical quantity χ, a special object for the value of the physical quantity and a special subobject E from the state object x. Since we need morphisms from one or more state(s) to another, if we were to have a category, we would need exponentiation. Since we need to determine if $x \vdash E$, then we could use the subobject classifier. We can be much more explicit:

1. A physical quantity x is represented by a morphism $\chi \in Hom_{\mathcal{C}}(X, R)$ of a universal object X (used to represent a state object) and R is a quantity-value object

2. Propositions p about the physical system modelled by X are represented by the subobject P, an element of the Heyting algebra, Ω^X. P is called a pseudo-state[9]

3. $x \vdash E$ if and only if $\chi_P \circ \widehat{x}$ is true, where E is a subobject of R.

For a more philosophical justification, see [2][6].

[9]By the Kochen-Specker theorem, there are no global elements of P in the particular topos of Hilbert Spaces, hence the name.

4 Future Work

In my talk, I did not talk about dynamics of a given physical system. There is no simple category theoretic version of time evolution that I know of. Tying up theoretical loose ends in my knowledge of the detailed relationship of category theory and physics is my next step. Later on, the relation of a topos to gauge theory needs to be investigated.

5 Appendix A: Uncertainty Principle

Definition 3 *Let $\hat{A}$,$\hat{B}$ be two self-adjoint operators on $\mathcal{H}$. The* ***commutator*** $\left[\hat{A},\hat{B}\right]$ *of $\hat{A}$,$\hat{B}$ is defined as $\hat{A}\hat{B}-\hat{A}\hat{B}$ and the anti-commutator $\left\{\hat{A},\hat{B}\right\}$ is defined as $\hat{A}\hat{B}+\hat{A}\hat{B}$*

An interesting consequence of these properties are the uncertainty relations, from which stems Hiesenberg's uncertainty relation

Definition 4 ***Deviation*** *$\triangle\hat{A}$ of an operator $\hat{A}$ is defined as*

$$\triangle\hat{A}=\hat{A}-\langle\psi,\hat{A}\psi\rangle\hat{I}$$

where $\left\langle\hat{A}\right\rangle=\langle\psi,\hat{A}\psi\rangle$ denotes the expectation value of $\hat{A}$ where $\psi\in\mathcal{H}$ is a state (that is, $\|\psi\|=1$). The

Lemma 5 *If $\hat{A}$ is self-adjoint, then so is $\triangle\hat{A}$*

Proof. $\hat{A}^*=\hat{A}$ implies $\triangle\hat{A}^*=\hat{A}^*-\overline{\langle\psi,\hat{A}\psi\rangle}\hat{I}^*=\hat{A}-\langle\hat{A}\psi,\psi\rangle\hat{I}=\hat{A}-\langle\psi,\hat{A}\psi\rangle\hat{I}$ ■

Lemma 6 $\left\langle\left(\triangle\hat{A}\right)^2\right\rangle=\langle\hat{A}^2\rangle-\langle\hat{A}\rangle^2$

Proof. From the definition of $\triangle\hat{A}$, it follows that

$$\begin{aligned}\left(\triangle\hat{A}\right)^2 &= \left(\hat{A}-\left\langle\hat{A}\right\rangle\hat{I}\right)^2\\ &= \left(\hat{A}-\left\langle\hat{A}\right\rangle\hat{I}\right)\left(\hat{A}-\left\langle\hat{A}\right\rangle\hat{I}\right)\\ &= \hat{A}^2-\left\langle\hat{A}\right\rangle\hat{A}-\hat{A}\left\langle\hat{A}\right\rangle+\left\langle\hat{A}\right\rangle^2\hat{I}\\ &= \hat{A}^2-2\left\langle\hat{A}\right\rangle\hat{A}+\left\langle\hat{A}\right\rangle^2\hat{I}\end{aligned}$$

Now, $\left\langle\left(\triangle\hat{A}\right)^2\right\rangle=\langle\psi,\left(\triangle\hat{A}\right)^2\psi\rangle=\langle\psi,\left(\hat{A}^2-2\hat{A}\left\langle\hat{A}\right\rangle+\left\langle\hat{A}\right\rangle^2\right)\psi\rangle$

$=\langle\psi,\hat{A}^2\psi\rangle-2\left\langle\hat{A}\right\rangle\langle\psi,\hat{A}\psi\rangle+\left\langle\hat{A}\right\rangle^2\langle\psi,\psi\rangle$

$= \langle \psi, \hat{A}^2 \psi \rangle - 2 \left\langle \hat{A} \right\rangle^2 + \left\langle \hat{A} \right\rangle^2$
$= \langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2 \quad \blacksquare$

Using this, we can define the **uncertainty** ΔA of $\hat{A}$: $\Delta A = \sqrt{\left\langle \left(\triangle \hat{A} \right)^2 \right\rangle} = \sqrt{\langle \hat{A}^2 \rangle - \langle \hat{A} \rangle^2}$. And now, for the celebrated Uncertainty Relation

Theorem 7 *Let $\hat{A}, \hat{B}$ be any two Hermitian operators. Then,* $\Delta A \triangle B \geq \frac{1}{2} \left| \left\langle [\hat{A}, \hat{B}] \right\rangle \right|$

Proof. If $\left(\triangle \hat{A} \right) \psi = \left(\hat{A} - \left\langle \hat{A} \right\rangle \hat{I} \right) \psi = \chi$ and $\left(\triangle \hat{B} \right) \psi = \left(\hat{B} - \left\langle \hat{B} \right\rangle \hat{I} \right) \psi = \phi$, then
$\langle \left(\triangle \hat{A} \right)^2 \rangle \langle \left(\triangle \hat{B} \right)^2 \rangle = \langle \psi, \left(\triangle \hat{A} \right)^2 \psi \rangle \langle \psi, \left(\triangle \hat{B} \right)^2 \psi \rangle$
$= \langle \chi, \chi \rangle \langle \phi, \phi \rangle$
$\geq \left| \langle \chi, \phi \rangle \right|^2$ by the Cauchy-Schwarz inequality
$= \left| \left\langle \left(\triangle \hat{A} \right) \psi, \left(\triangle \hat{B} \right) \psi \right\rangle \right|^2$
$= \left| \left\langle \psi, \left(\triangle \hat{A} \triangle \hat{B} \right) \psi \right\rangle \right|^2 = \left| \langle \triangle \hat{A} \triangle \hat{B} \rangle \right|^2$

From a direct calculation, it can be inferred $\left| \left\langle \triangle \hat{A} \triangle \hat{B} \right\rangle \right|^2 = \frac{1}{4} \left| \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle \right|^2 + \frac{1}{4} \left| \left\langle \left\{ \hat{A}, \hat{B} \right\} \right\rangle \right|^2$

$\Rightarrow \left| \left\langle \triangle \hat{A} \triangle \hat{B} \right\rangle \right|^2 \geq \frac{1}{4} \left| \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle \right|^2$

Since $\left\langle \left(\triangle \hat{A} \right)^2 \right\rangle \left\langle \left(\triangle \hat{B} \right)^2 \right\rangle \geq \left| \left\langle \triangle \hat{A} \triangle \hat{B} \right\rangle \right|^2$ and $\left| \left\langle \triangle \hat{A} \triangle \hat{B} \right\rangle \right|^2 \geq \frac{1}{4} \left| \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle \right|^2$

we have $\Delta A \Delta B \geq \frac{1}{4} \left| \left\langle \left[\hat{A}, \hat{B} \right] \right\rangle \right| \quad \blacksquare$

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